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FANO MANIFOLDS – OLD AND NEW

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(with an appendix by Victor Przyjalkowski)

Algebraic manifolds and Fano manifolds

By a theorem of d'Alembert (1746) any polynomial $f(x)$ with complex coefficients of one variable has always a complex root, i.e. complex number c such that $f(c) = 0$. If $deg(f) = d$ then the number of roots with multiplicities is d, with general choice of the coefficients of f the d roots are different. This is generalized for (complex) polynomials in $n \geq 2$ variables: The set of solutions of $f(x_1, ..., x_n) = 0$ is a hypersurface of degree d, and for general choice of coefficients this hypersurface is smooth and irreducible.

Two polynomials of $n \geq 2$ variables may not always have common solutions: For example $f = x^2 + y^2 - z^2 - 1 = 0$ and $g = x^2 + y^2 - z^2 - 2 = 0$ have no common solution in the complex space C^3. But in the projectivized space $P^3(x : y : z : w)$ the projectivized equations

$$F = x^2 + y^2 - z^2 - w^2 = 0 \text{ and } F = x^2 + y^2 - z^2 - 2w^2 = 0$$

always have a common solution; in the case this is the smooth conic curve

$$\text{C: } F - G = w^2 = 0,\ 2F - G = x^2 + y^2 - z^2 = 0.$$

From more algebraic point of view, i.e. as a scheme, this curve has to be taken with multiplicity two because of $w^2 = 0$. To get the reduced curve one has to replace the ideal (F, G) with its radical $(w, 2F - G)$.

Three projective polynomials on $\mathbf{P}^3(x : y : z : w)$ have at least finite common set of solutions. For example the set C of solutions of $xz - y^2 = xw - yz = yw - z^2 = 0$ is the space cubic curve $(x : y : z : w) = (1 : t : t^2 : t^3) : t \in \mathbf{C} \cup \infty \cong \mathbf{P}^1$.

Also one can take in mind that e.g. $xy = xz = 0$ have as a common solution the line $x = 0$ plus the point $(x : y : z) = (1 : 0 : 0)$.

A (complex) projective variety is any common set $X \subset \mathbf{P}^n$ of solutions of a finite number of projective (i.e. homogeneous) polynomials $(F_1, .., F_s)$. Algebraic operations (e.g. taking radicals and saturations) can make this and further definitions more precise, see e.g. [Har1977] The manifolds are the smooth varieties, and the questions is to find some idea about their various descriptions, for example how to classify them by some invariants common for all of them.

The first is by topology: Any smooth irreducible complex projective variety (a complex manifold) X, say of dimension n is also a real differentiable manifold X of real dimension $2n$, which is always orientable, i.e. the transition functions from chart to chart have always positive Jacobian determinant.

First classification invariant of complex manifolds is the dimension, and the next invariants of complex manifolds of dimension n can be e.g. their homology groups $H_k(X, \mathbf{Z})$, $0 \leq k \leq 2n$ as real differentiable manifolds.

For example smooth algebraic curve C is a real differentiable surface, which is orientable (i.e. $H_2(C, \mathbf{Z}) = \mathbf{Z}$, and further properties tell that the homology $h_1(C) = 2g$ is always even, where g is called the genus of C.

Any projective manifold is also a complex manifold, i.e. an analog of differentiable manifold, with holomorphic transition functions.

From this point of view, the (smooth) complex n-fold X has a homomorphic tangent bundle T_X and its dual bundle Ω_X of holomorphic 1-forms. If Ω^k are the vector bundles (or the locally free sheaves) of holomorphic k-forms on the smooth complex n-fold X, the vector bundle Ω^n of differential n forms on X is of rank 1, and the transition functions of Ω^n are the (reciprocal to the) Jacobian functions of the transition matrices for the differentiable manifold X. Being a rank 1 bundle on the smooth manifold X, Ω^n is locally generated by one meromorphic function on X, and the zeros of this locally meromorphic function on X form an algebraic cycle [1] $K = K_X$ of X which is either empty or has dimension $n-1$. This set is well defined on X, in sense that different choices (say 1 and 2) of local holomorphic atlases on X define as above two different canonical cycles K_1 and K_2 (as zeros and poles of locally meromorphic functions on X), but the difference $K_1 - K_2$ is always the cycle of zeros and poles of a globally meromorphic function on X, which by definition means that K_1 and K_2 belong to the same class of divisors on X, that is called the canonical class K_X of X.

The zeros and poles of any locally meromorphic function on the n-fold X (i.e. the Weil divisors on X) are integer linear combinations of submanifolds of X of dimension $n-1$. If in the class K_X there exists a divisor K with only negative coefficients, then K is called negative. A particular case of negative canonical class is the case when K_X is ample.

What is an ample divisor on a projective manifold X ? Since X is projective then $X \subset \mathbf{P}^N$ as a set of zeros for some projective space P^N, but this embedding is far from unique, for example, $X \subset P^N$ can be reembedded by polynomials of degree d in other projective space $\mathbf{P}^N$ and then one can project from a subspace far from the image of X. The theory gives that all projective maps are given by this way. A divisor D on X is called very ample on X if in some of these embeddings of X it happens that D and the hyperplane section of X belong to the same class. A divisor D is ample if some multiple of it is very ample. For example is x is a point on the plane cubic curve C then x is ample since $3x$ is very ample. In fact also x, y, z are three points on C then the divisor $x+y-z$ of degree 1 is also ample. On a curve any divisor of positive degree is ample.

This is not the same e.g. for surfaces (and therefore for manifolds of higher dimension). For example $P^1(s_0 : s_1) \times P^1(t_0 : t_1)$ embeds in $P^3(x, y, u, v)$ by $(s_0t_0 : s_0t_1 : s_1t_0 : s_1t_1)$ as a quadric surface S with the equation $xw - yz = 0$.

The image $L = (t_0 : t_1 : 0 : 0)$ of the line $(1 : 0) \times P^1(t_0 : t_1)$ is a positive divisor (in fact a line) on s which is not ample

The hyperplane class of S is $H = L + M$ where $S = (0 : 0 : s_0 : s_1)$.

One can compute that the canonical class $K_S = -2L - 2M$, and therefore $-K_S$ is ample (as two times an ample divisor), and that the ample divisors on S are $D = aL+bM$ with $a > 0, b > 0$; on this particular surface all ample divisors are already very ample.

[1] i.e. a sum of algebraic subvarieties with positive or negative coefficients, or zero

Definition and examples of Fano manifolds

Definition. The projective manifold X is Fano [2] if the class $-K_X$ is ample.

For general references about Fano manifolds - see [IP1999], [Deb2001]; the biography of Gino Fano by Alessandro Verra – in [arxiv:1311.7177] There exist generalizations for certain classes of singular varieties, see e.g. [math.AG:9905068] and other papers of H. Takagi on the classification of Q-Fano 3-folds.

Examples:

<u>In dimension 1</u> the only Fano manifold is the projective 1-space $\mathbf{P}^1$

<u>In dimension 2</u> the only Fano manifolds are $\mathbf{P}^2$, the quadric surface $\mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^3$, and the del Pezzo surfaces $S_d, 1 \leq d \leq 8$, for del Pezzo surfaces - see e.g. the book [Yu.Manin, Cubic Forms].

Why e.g. the "del Pezzo" type surface $S = \mathbf{P}^2$ blown up at $n \geq 9$ points is not Fano ? Let e.g. $n = 9$, and let $S = \mathbf{P}^2$ blown up at 9 general points $p_1, .., p_9$. The anticanonical class $-K_S$ = the cubic curves through the 9 points, and therefore $(-K_S)^2 = 0$. If $-K_S$ is ample then $-mK_S$ is very ample, and then $(-mK_S)^2 > 0$ (because $H = -mK_S$ is a hyperplane section of S embedded in some projective space P^N. However then $(-mK_S)^2 = H^2 = deg(S) > 0$. But $(-mK_S)^2 = m^2K_S^2 = 0$ – contradiction.

<u>In dimension 3</u> there are 105 projective families of smooth Fano threefolds.

They belong to two classes: 17 classical families with 2-nd Betti number $B_2 = 1$ and 88 families with $B_2 \geq 2$.

Almost all of the classical families are found by G. Fano (1930, 1937) and L. Roth (1955), and then the list was completed by V.Iskovskikh, see [Math. USSR, Izv., Vol.11:3 p.485-527 (1978); Vol.12:3 p. 469-506 (1978)], with one exception added by Mukai and Umemura, see [LN Math. 1016, p.490-518 (1983)].

The 17 families of smooth Fano threefolds with $B_2 = 1$ are:

The projective 3-space P^3; The quadric threefold Q_2;

The 5 Fano's Y with $B_2 = 1$ of index 2, i.e. $K_Y = -2H$, where $Pic(X) = \mathbf{Z}H$ are:

$Y_1 :=$ the smooth degree 6 hypersurfaces in the weighted projective space $P^4(1, 1, 1, 2, 3)$ (Veronese double cone);

Y_2: = the smooth degree 4 hypersurfaces in $P^4(1, 1, 1, 1, 2)$ (quartic double solid)

$Y_3 \subset P^6$ the cubic threefold; $Y_4 = Y_{2.2} \subset P^5$ the complete intersection of two quadrics;

$Y_5 = G(2, 5) \cap P^6$ (the del Pezzo Fano threefold)

The 10 Fano's X with $B_2 = 1$ of index 1, i.e. $K_X = -H$ (see above) are:

$X_2 \subset P^4(1, 1, 1, 1, 3)$ (the sextic double solid)

$X_4 \subset P^4$ the quartic threefold $X_6 = X_{2.3} \subset P^5$, $X_8 = X_{2.2.2} \subset P^6$,

the threefolds $X_{10}, X_{12}, X_{14}, X_{16}, X_{18}$ and X_{22}, see e.g. [Muk1995].

The other 88 families, i.e. the smooth Fano threefolds with $B_2 \geq 2$ are added by Mori and Mukai (87 families in 1981, and one more in 2003), see [Manuscr. Math. Vol.36 p.147-162 (1981); addendum + erratum: ibid. Vol.110 p.407 (2003)], or the lists at the end of the book [IP1999].

Here are examples of some of them: $P^2 \times P^1$, a hyperplane section of $P^2 \times P^2$, blowup of P^3 at an elliptic curve of degree 4; blowup of P^3 at a disjoint union of two lines,

[2] Gino Fano - 1871 Mantua, Italy – 1952 Verona, Italy

$P^1 \times S_d$, where S_d is a del Pezzo surface of degree $1 \leq d \leq 6$, etc.; the addendum in 2003 is : Blowup of $P^1 \times P^1 \times P^1$ at a smooth (rational) curve of tridegree (1,1,3) (a Fano 3-fold with with $-K^3 = 26$, $B_2 = 4$, $B_3 = 0$).

Classical results and approaches

For reviews on the classical results – see e.g. [Bea1977] and [Isk1983].

The Lüroth problem

In 1876 Lüroth has proved that if $\mathbf{C} \subset L \subset \mathbf{C}(x)$ are inclusion of fields with $L \neq \mathbf{C}$ then L is isomorphic to $\mathbf{C}(x)$.

For example $\mathbf{C} \subset \mathbf{C}(x^7) \subset \mathbf{C}(x)$ and then we let $y = x^7$, and $\mathbf{C}(x^7) = \mathbf{C}(y)$; or otherwise we let $x^7 \mapsto x$ and this yields $\mathbf{C}(x^7) \cong \mathbf{C}(x)$.

The Lüroth problem (for a review, including on on-closed fields and fields of non-zero characteristic – see [Wil06]: The problem is whether the same is true for n variables, i.e. whether for algebraic n-folds X the existence of a dominant rational map $\mathbf{P}^n \to X$ (=: X is unirational) implies the existence of other again dominant rational map $\mathbf{P}^n \to X$ which has an inverse rational map (i.e. of generic degree 1) (=: X is rational)

For example $L = C(x^2, y^6) \subset C(x, y)$ and after replacing $x^2 = s, y^2 = t$ we get $L \cong C(s,t)$. Less-trivial example: the field $L = C(S)$ of the Fermat cubic surface

$$S : x^3 + y^3 + z^3 = 1.$$

The quotient field $L = C(S)$ is the quotient field of the ring $C[S] = [x, y, z]/(x^3 + y^3 + z^3 = 1$ of S; the last is an integral domain since S is smooth and irreducible (i.e. by some of the Hilbert theorems). The field $L = C(S)$ is embedded in $C(s,t)$ by:

$x(s,t) = \frac{9t-(s^2+st+t^2)^2}{3t(s^2+st+t^2)-9}$, $y(s,t) = \frac{9s+9t+(s^2+st+t^2)^2}{3t(s^2+st+t^2)-9}$, $z(s,t) = \frac{-9-3(s^2+st+t^2)(s+t)}{3t(s^2+st+t^2)-9}$,

and therefore by the Castelnuovo's criterion the Fermat cubic surface is rational. [3]

All smooth cubic threefolds are unirational. The first examples of unirational smooth quartic 3-folds belong to B.Segre (1960) [4]. It is still not known whether the general smooth quartic threefold is unirartional, but in fact, there are not known examples of non-unirational Fano threefolds. Unirationality is a not very well studied area; some more relatively new results – by M. Marchisio and N.F. Zak.

The following example (perhaps coming from B. Segre) is taken from 'Degrees of Irrationality of Fano manifolds' of I. Cheltsov (in Russian):

The quartic $x_0^4 + x_o x_4^3 + x_1^4 - 6x_1^2 x_2^2 + x_2^4 + x_3^4 + x_3^3 x_4 = 0$ is smooth and unirational, i.e. there exists a finite map $\mathbf{P}^4 \to X$, but there does not exist a birationality $\mathbf{P}^4 \leftrightarrow X$.

For curves: unirational = rational – answered by Lüroth - 1876.

For surfaces: unirational = rational – by a rationality criterion for surfaces of Castelnuovo (1893) – a solution of the Lüroth problem for dimension = 2.

[3] In fact any smooth complex cubic surface is rational, and also any singular cubic surface is rational, except the cone over a smooth plane cubic curve. However: see an example by J. Kollar of a non-rational real smooth cubic surface and related questions in [math.AG:9707013].

[4] Beniamino Segre 1903–1960; Corrado Segre 1863–1924

For threefolds: By a result first announced by G. Fano (1907-8; 1915) any smooth quartic threefold X is non-rational, because Bir(X) = Aut(X). The approach proposed by G. Fano had used earlier ideas of M. Noether and new ideas of Fano. Later the proof of Fano had been criticized as non-complete by L. Roth (in the 1950's), and the proof was completed in the paper 'Three-dimensional quartics and counterexamples to the Lüroth problem' by V.Iskovskikh and Yu.Manin in [Math.USSR, Sbornik 15:1 (1971) p.141-166).

At the same time M.Artin and D.Mumford in [Proc. Lond. Math. Soc. (3) 25 (1972), 79-95] give an example of other unirational but not-rational Fano threefold – this is a special double covering X of $\mathbf{P}^3$ ramified over a singular quartic surface. They show that the 3-rd integer homology group H_3 of X is $\mathbf{Z}_2$, and because this group for a threefold is a birational invariant (observed and proved again by Artin and Mumford) then X is non-rational.

And again at the same time Clemens and Griffiths in [The intermediate Jacobian of a cubic threefold, Annals of Mathematics, 2nd Ser., Vol. 95, No. 2. (Mar., 1972), pp. 281-356] show that any smooth cubic threefold X(3) is unirational but non-rational. For the unirationality of X(3) they write (no reference) that it is first seen by Max Noether and write there a unirationality construction of X(3) by Fogarthy.

The method of H.Clemens and Ph.Griffiths uses the difference in the cohomology between (in the case) the cubic threefold $X = Y_3$ and the projective space $\mathbf{P}^3$. First on any complex compact Kähler manifold (in particular algebraic) can be defined additional decompositions of the cohomology spaces $H^k(X, \mathbf{C})$ into subspaces $H^{p,q}, p + q = n$.

For example $\mathbf{P}^3$ has as only non-zero Hodge numbers $h^{k,k} = 1$, $k = 0, 1, 2, 3$ while the cubic threefold has as only non-zero Hodge numbers the same $h^{k,k} = 1$, $k = 0, 1, 2, 3$ and also $h^{2,1} = h^{1,2} = 5$.

Not all of the dimensions $h^{p,q} = \dim H^{p,q}$ are birational invariants. but e.g. the genera $h^{k,0}$ are birational invariants.

For example for the quintic threefold $h^{3,0} = 1$ and therefore the quintic threefold is not rational. Yes but for the cubic 3-fold all $h^{k,0}$ are zero as for the projective 3-space.

Clemens and Griffiths use additional structures connected with the spaces $H^{2,1}$ and $H^{1,2}$ which are also birational invariants. Let tell that $h^{2,1}$ of a threefold is not a birational invariant: For example the blow-up $X \to P^3$ along an elliptic quartic curve (an intersection of two quadrics) is one of the Fano 3-folds of $b_2 = 2$ (No.25 in the tables of Fano 3-folds on p.215-225 of [IP1999]) with $K^3 = 32$ and with $h^{2,1} = 1$, while the blow-up is a birationality.

The additional structure by Clemens and Griffiths is the 5-dimensional torus $J(X) = H^{2,1}(X)^*/H_3(X, \mathbf{Z})$ (modulo torsion). Similar torus, the Jacobian of a curve, had been studied earlier. These are algebraic principally polarized tori, which means than on them are well defined unique (up-to translation) theta-functions. The zero-sets of theta functions are divisors (hypersurfaces) Θ in $J(X)$, and if $J(X)$ does not contain a non-zero Jacobian of a curve as a direct summand, then $J(X)$ and therefore the variety Θ are birational invariants. Clemens and Griffiths prove that indeed the Jacobian of a smooth cubic threefold $J(X)$ cannot be a sum of Jacobians of curves and therefore X is non-rational.

This method had been developed to show that several other Fano threefolds as well some other higher-dimensional algebraic manifolds are non-rational.

Results since the 70's and the 80's

Rationality questions: Keep the above notation P^3, Q_2, Y_d, X_d of the 17 Fano threefolds with $B_2 = 2$.

For 8 of these Fano's: $\mathbf{P}^3$, the quadric Q_2, the complete intersections of two quadrics Y_4, the del Pezzo threefold Y_5 = any smooth complete intersection of the Grassmannian $G = G(2,5) \subset \mathbf{P}^9$ and a subspace $\mathbf{P}^6$, and the threefolds $X_{2g-2} \subset \mathbf{P}^{g+1}, g = 7, 9, 10, 12$ (as well the exceptional X'_{22} - see [MU], [F]), any threefold is rational. This follows directly from the geometric descriptions of these varieties

The Clemens-Griffiths approach (CG) yields that the general element of the other 10 families is non-rational, see [Bea1977].

By the approach (FI) of Fano-Iskovskikh is proved that any smooth Fano threefold Y_1, X_2 and X_4 is non-rational, and again that the general X_6 is not rational (see [Iskovskikh V., Birational automorphisms of three-dimensional algebraic varieties, J. Soviet Math. vol.13 (1980) 815-868]

More recent results by the classical approaches

Some of these results are reproved for certain mildly singular Fano threefolds. For example:

– the general Y_2 with less than 5 nodes is also non-rational – by (CG) approach by [Debarre]

– a sextic double solid with 14 or less nodes is non-rational – by (FI) approach by I. Cheltsov and J. Park, see [math.AG:0404452]

– a quartic threefold X_4 with 8 or less nodes is non-rational – by (FI) approach by I. Cheltsov (Pac.J.Math. 226:1 (2006))

More can be read in the review paper of I. Cheltsov "Birationally rigid Fano varieties", Russ. Math. Surv., 60:5, p.875–965 (2005)

More new result (by (FI) approach): I. Cheltsov and J. Park, see [arxiv:1309.0903]. Proves birational rigidity and hence non-rationality of a long list of mostly singular weighted Fano hypersurfaces.

These hypersurfaces are the 95 families $X_d \subset P^4(w_1, w_2, w_3, w_4, w_5)$ with w_i weights and d degrees classified by Iano and Fletcher in the 90's. These are the Fano weighted hypersurfaces of index one with certain restriction on the singularities (this excludes e.g. the hypersurfaces of degree 1,2,3)

The Iano-Fletcher list starts with $X_4 \subset P^4(1,1,1,1,1)$, $X_5 \subset P^4(1,1,1,1,2)$, $X_6 \subset P^4(1,1,1,2,2)$, ..., around the middle $X_{21} \subset P^4(1,1,3,7,10)$, ... until the end with $X_{42} \subset P^4(1,3,4,14,21)$ and $X_{50} \subset P^4(1,3,5,16,24)$.

Cheltsov and Park prove that all these 3-dimensional weighted Fano hypersurfaces (of index 1) are birationally rigid, and therefore non-rational.

In higher dimensions for non-rationality still can work only the (FI) method, besides of course topological methods, that however cannot be applied for Fano varieties because the birational invariant cohomology of these varieties are the same as for the projective space. [5]

[5] Again with an exception - the 3-rd torsion group for threefolds: after the Artin-Mumford example there are two other examples of Fano 3-folds with torsion in the 3-rd homology.

– Any smooth Fano hypersurface $X_N \subset P^N$ for $N \geq 5$ is birationally rigid and therefore non-rational, see [A.V.Pukhlikov: math.AG:0201302]; the result for N=5 is earlier – since the end of the 80's); the same is true also for the same kind general Fano hypersurfaces with isolated nodes in general position, see [A.V.Pukhlikov: math.AG:0106110]

– The general Fano complete intersection $X_{d_1,...,d_k} \subset P^{M+k}$ (all $d_i \geq 2$) of index 1 (i.e. $d_1 + ... + d_k = M + k$) are birationally rigid (and therefore non-rational) for $M \geq 2k + 1$) – see [J. reine angew. Math. 541 (2001), 55–79]

– Ana-Maria Castravet: negative answer (arxiv:0604548) to a question by Pukhlikov: Is any smooth Fano variety of dimension ≥ 4 and index 1 is birationally rigid ?

– A. Beauville: "Non-rationality of the symmetric sextic Fano threefold" $X_6 = X_{2.3} \subset P^6$, see [arxiv: 1102.1255]; due to a non-filled gap in the (FI) approach (in this case) non-rationality for X_6 is proved only for general varieties (by both approaches). Beauville shows that the Jacobian torus of the symmetric threefold

$$X_6 = (\Sigma x_i = \Sigma x_i^2 = \Sigma x_i^3 = 0) \subset \mathbf{P}^7$$

is not a sum of Jacobians of curves - since it has more automorphisms (= 2520) than any sum of jacobians of curves of the same dimension (equal to 20 in the case), the maximum order of the last is 84(g-1) = 84(20-1) = 84.19 = 1596 which is less than 2520.

– For more results on rationality and unirationality see:

Smith K. Rational and Non-Rational Algebraic Varieties (Lectures of Janos Kollar) math.AG:9707013

Kollar J. Unirationality of cubic hypersurfaces 2002,

Kollar J. Nonrational hypersurfaces, Journ.A.M.S. 1995

Questions:

(1) What about non-rationality of weighted Fano hypersurfaces $X_N \subset P^N(w_0, ..., w_M)$ of index one in any dimension ≥ 3 ? (following e.g. Pukhlikov and Cheltsov-Park)

(2) The same question – for weighted Fano complete intersections (as above) of index one in any dimension ≥ 3: (following Pukhlikov)

(3) One should also find a similar full or partial analogs of the Iano-Fletcher list also for weighted Fano hypersurfaces in dimensions ≥ 4: yes, these will be long lists ...

Results about moduli of vector bundles and derived categories

In the 70's appear a sequence of papers on descriptions of moduli of vector bundles on the projective spaces: e.g. [on P^2: Barth W., Invent. Math. 42 (1977) 63-91], [(on P^3): Hartshorne R., Math. Ann 238 (1978), p. 229-280]

This in particular states the question to study vector bundles, and more general sheaves and complexes of sheaves on Fano manifolds, starting from the projective line, Fano surfaces and Fano threefolds.

As for the projective line P^1, any vector bundle on P^1 splits into a sum of line bundles $O(d_i)$, but this is no more true for P^2, e.g. the trank two tangent bundle T_{P^2} is indecomposable. By a criterion of Horrocks [Proc. London Math. Soc. 14 91964), 689-713] any vector bundle E on P^n with no intermediate cohomology i.e. ($H^i(P^n, E(d)) = 0$ for all $0 < i < n$ and all d; call these for short ACM vector bundles) splits intio a sum of line bundles, and one can state the question how and when this criterion extends e.g. for Fano manifolds as the closest algebaric varieties to projective spaces.

As earlier papers on the subject (for the questions regarded below) one can mention: [Ottaviani G. Some extensions of Horrocks criterion ... on grassmannians and quadrics, 1989] and [E. Arrondo, L. Costa, Vector bundles on Fano 3-folds (of index 2) without intermediate cohomology, math.AG:9804033] where in particular are classified all (non-decomposable) rank 2 ACM vector bundles on Fano 3-folds of index 1.

For Fano threefolds of index 1 these questions has been stated and initiated to be studied for some particular ACM rank two vector bundles by D.Markushevich and A.Tikhomirov, followed by A.Iliev and D.Markushevich, St. Druel, C.Madonna, etc. (for a more conceptual point of view on these questions see the two papers of A. Beauville [math.AG:9910030], [math.AG:0005017]). For ACM bundles on del Pezzo surfaces one can give an example reference [Faenzi D., Rank 2 ACM bundles on cubic surfaces: arxiv.math:0504492].

In the paper [arxiv.math:0103010] of C. Madonna is conjectured the complete list of all possble non-decomposable rank two ACM vector bundles on Fano threefolds of index 1. The most of the bundles in this list (containing around 100 possible bundls) exist by relatively simple reasons, but the existencde of all remained a question. The existence of all the bundles from the Madonna's list has been proved later by M.C. Brambilla, D. Faenzi and E. Arrondo, together with a stidy of the moduli of some of them, see e.g. [math.AG:0806.2265]. The moduli of other particular bundles from this list have been studied at the same time by D.Markushevich, A. Iliev, L. Manivel, K.Ranestad, and more recently by G.Kapustka and K.Ranestad (see [arxiv.math:1005.5528]).

Question: Notice that in these papers are studied mostly rank two ACM vector bundles on Fano threefolds with $B_2 = 1$ of index one. One can state therefore the question to study (in particular to classify, and then to sudy their moduli) of higher rank ACM vector bundles on these Fano threefolds.

Particular examples of such higher rank bundles of higher rank on Fano threefolds of index two have been studied in the cited above paper of Arrondo and Costa, as well in some of the other papers cited above. For example, the Mukai classification of Fano threefolds with $B_2 = 1$ of index 1 is based also on existence of some rigid (i.e. with no moduli) vector bundles of rank 2 or 3 on these varieties, see [Muk1995].

In more modern perspectuive, study of vector bundles on e.g. a variety X is substituted by a study of the derived category $Der(X)$ generated by complexes of coherent sheaves on X. An introduction text: [A. Caldararu, Derived categories of sheaves: a skimming].

By a theorem of Beilinson (in [Functional Analysis and Appl. 12 (1978) p.214-216]) the bounded derived category $D(P^n)$ is generated by the exceptional sequence of sheaves $(O(-n), O(-n+1), ..., O(-1), O)$, for a proof see the book [C.Okonek, M.Schneider, H.Spindler, Vector bundles on complex projective spaces, 1980].

One can state the question how this result can be extended to other algeraic varieties. It turns out that such exceptional sequence is seen to exist only for certain varieties close to projective spaces: see [M. Kapranov, Derived category of coherent sheaves on Grassmann manifolds (1983)], [A. Bondal, D. Orlov, Semiorthogonal decompositions for algebraic varieties, math.AG:9506012], [Bohning C. Derived categories of coherent sheaves on rational homogeneous manifolds, arxiv:0506429] until the newest review of A. Kuznetsov: [Kuznetsov A., Semiorthogonal decompositions in algebraic geometry arxiv:1404.3143]

It turns out that such exceprional collections (semiorthogonal decompositions) are even not known for all homogeneous manifolds. As an example the relatively recent paper [arxiv: 0910.2356] and of A.Polishchuk and A. Samokhin shows the existence of such decompositions forthe derived categories of the two lagrangian grassmannians LG(4,8) amd LG(5,10).

Nevertheless, for some of the simple Fano threefolds the derived category is known – for P^3, the quadric Q_2 and also (by results of A.Kuznetsov and D. Faenzi since the 90's) for the derived categories of the Fano threefolds Y_5 and X_{22}.

Question: Looking from an other perspective the four Fano threefolds P^2, Q_2, Y_5 and X_{22} with an exceptional collection generating their derived categories are also all the smooth projective compactifications of the complex 3-space $\mathbf{C}^3$, see e.g. [Furushima M., Compos. Math. 76:2 (1990) p.163-196]. The derived category of the quadric Q_2 is generated by O(1) and the spinor rank 2 bundle (with $c_1 = -1, c_2 = 1$).

Therefore one can state the question whether one can find similar semiorthogonal decompositions (is such exist) also for certain smooth projective compactifications of affine spaces $\mathbf{C}^n$ in higher dimesnions.

Although examples of such compactifications different from the projective space P^n (e.g. the n-dimensional smooth quadrics) can be shown in any dimension, as far as I am informed, a complete classification of these compactifications is not known even in dimension 4. In dimension 4, particular examples of such compactifications have been studied e.g. by Yu. Prokhorov, see [Prokhorov Yu.G. Compactifications of C^4 of index 3, Proceedings Yaroslavl' 1992 (1994)]. One should however mention that the above question seem not to be trivlal, since e.g. all the Grassmannians are projective compactifications of complex spaces, but as seen from the previous comments, semiorthogonal decompositions are known not for all Grassmannians. But for lower dimensions the question seem to be non-empty in advance.

At the end, one more problem:

Smooth Fano manifolds of dimensions 4 (or more) are not completely classified. There are particular constructions of Fano 4-folds, and it seems by projective geometry one can get more. For example, in the paper [Math. Z. 218, 563-575 (1995)] of Oliver Küchle are found the Fano 4-folds of index 1 which are zero-sets of sections of homogeneous vector bundles over Grassmannians. It is possibhle that there may exist also other still not described Fano fourfolds of index one in other homogeneous varieties, and constructed by similar ways, say in the homogeneous varieties of the exceptional groups, or by higher scheme techniques. As an example one can take some descriptions of surfaces and threefolds of low codimension, constructed "via commutative algebra [6] methods":

W. Decker, S.Popescu, Threefolds in P^5 and surfaces in P^4, math.AG:9402006
Walter Ch., Pfaffian-subschemes, alg-geom:9406005
Tonoli F., Construction of Calabi-Yau 3-folds in P^6, J.Alg.Geom. 13:2 (2004) 209232
Marie-Amelie Bertin, Examples of Calabi-Yau 3-folds of P^7, arxiv:0701511
Kapustka G. & M., A cascade of determinantal Calabi-Yau threefolds, arxiv:0802.3669

[6]taken from the abstarct of the cited here paper [arxiv:0701511]

Homological Algebra and Mirror Symmetry

Appendix by Victor Przyjalkowski

Another, new approach to studying Fano varieties is related to Homological Algebra and Mirror Symmetry. Derived and simplicial invariants can characterize smooth Fano varieties and their invariants. Say, according to Bondal–Orlov Theorem one, given a bounded derived category of coherent sheaves on a Fano variety, can reconstruct the variety itself. Another example is related to symplectic geometry. Considering a Fano variety as a symplectic manifold (in particular this means that we do not distinguish, say, two different smooth hypersurfaces of the same dimension and degree — all of them are symplectomorphic) one can associate a set of so called Gromov–Witten invariants. These invariants count curves lying on a Fano variety and define a Quantum Cohomology ring, a deformation of a usual cohomology ring. The Gromov–Witten theory is expected to differ families of Fano manifolds; no counterexamples are known.

Mirror Symmetry approach came to mathematics from physics a couple of decades ago. From mathematical point of view it states that for any object it considers, say, for a Fano variety, there exists a pair for which two geometries for an object, algebraic and sympolectic, interchanges for a dual object. For instance, it states that for any Calabi–Yau variety it should exist another Calabi–Yau whose Hodge diamond is the one for the initial Calabi–Yau rotated by 90 degrees; this justifies the term Mirror Symmetry. However the pair object for a Fano variety is not a variety again, but a so-called Landau–Ginzburg model — a variety equipped by a complex-valued function called superpotential. Given a Fano variety X one can associate two categories, $D^b(Coh-X)$ and $Fuk(X)$ that reflects algebraic and symplectic sides of X accordingly. In a similar way one can associate with a dual Landau–Ginzburg model $LG(X)$ two categories as well, $D^b_{sing}(LG(X))$ and $FS(LG(X))$. Homological Mirror Symmetry conjecture formulated by Kontsevich in 1994 states that for any (in our case Fano) variety X there should exist a Landau–Ginzburg model $LG(X)$ such that their categories are cross-equivalent: $D^b(Coh-X) \sim FS(LG(X))$ and $Fuk(X) \sim D^b_{sing}(LG(X))$. Being proven this duality enables us to study algebraic and birational invariants of X in terms of symplectic ones of $LG(X)$. For example in a spirit of Clemens–Griffits approach one can treat rationality of problem for X in terms of monodromy of family provided by $LG(X)$.

Landau–Ginzburg models were suggested for a large class of Fano varieties. The most famous construction, for complete intersection in smooth toric varieties, was suggested by Givental. Unfortunately, mirror duality for Fano varieties are (partially) proven only up to dimension 2 (by Auroux, Katzarkov, and Orlov). The problem is rapidly increasing complexity of symplectic geometry when one increase the dimension. To fix this problem one can weaken the definition of mirror duality taking some consequences of Homological Mirror Symmetry as a definition of a weakened duality. One of approaches of this type is a (strengthened) Mirror Symmetry of variations of Hodge structures. It states that the dual object for a Fano variety is just a specific Laurent polynomial called toric Landau–Ginzburg model. Periods of this polynomial encodes Gromov–Witten invariants of a Fano variety, and a Newton polytope of the polynomial provides a toric degeneration of a Fano variety. Such toric Landau–Ginzburg models were proven to exist (in some cases not all conditions are proven to hold) for rank one Fano threefolds (Przyjalkowski, Ilten–Lewis–Przyjalkowski), all smooth Fano threefolds (Coates–Corti–Galkin–

Golyshev–Kasprzyck, Doran–Harder–Lewis–Katzarkov–Przyjalkowski, Ilten–Kasprzyck–Katzarkov– Przyjalkowski–Sakovich), complete intersections in (weighted) projective spaces (Przyjalkowski (based on Givental's suggestions), Ilten–Lewis–Przyjalkowski), complete intersections in Grassmannians of planes (Przyjalkowski–Shramov). One can reconstruct some invariants of the initial Fano variety from the Laurent polynomials. Say, in a threefold case one can compute a dimension of intermediate Jacobian of a Fano variety (Przyjalkowski,Doran–Harder–Lewis–Katzarkov–Przyjalkowski); similar can be done for complete intersections (Przyjalkowski–Shramov). The explanation of this phenomena is done by Katzarkov–Kontsevich–Pantev where an analog for Landau–Ginzburg models of Hodge diamond rotation is suggested. Another example is treatment of rationality of a Fano threefold in terms of quasiunipotentness of a monodromy of a dual Landau–Ginzburg models. This is done by comparing Iskovskikh at al. studying of rationality of Fano threefolds and Golyshev's monodromy calculations.

Which Laurent polynomials can be dual to Fano varieties? It turns out that in appropriate basis there are strong restrictions on their coefficients. There are a series of suggestions what restrictions should be. Przyjalkowski suggested binomial principle saying that coefficients on edges of a Newton polytope of a Laurent polynomial should be certain binomial coefficients. Then London group (Coates, Corti, Galkin, Golyshev, Kasprzyck,...) suggested its generalization called Minkowski principle relating coefficients to Minkowski decomposition of facets. Finally they suggested maximally mutation principle that is related to mutations of polytopes. They studied all canonical Gorenstein toric threefolds, constructed all Laurent polynomials related to them that satisfied this principle, computed Gromov–Witten invariants of Fano threefolds, and related them with periods of Laurent polynomials. In this way they got exactly all Fano threefolds classified by Iskovskikh and Mori–Mukai. This extends Przyjalkowski proof of Golyshev conjecture from rank one case to all cases. The hope is to use this approach to make one more step to a classification of smooth Fano fourfolds.

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On the subject I can only add several references

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